Something I read a while ago, about constructing an approximation to a conservative system by performing the difference equation update in a *ping–pong* fashion, i.e. using speed to update position and then the new position to update speed, instead of performing the updates in parallel. Supposedly, this preserves the *symplectic structure*. I lost the reference, so this is an attempt to reconstruct it.

This might be related with approximations to 2nd order CT filters mentioned in Stilson's PhD thesis[1], where two integrators are replaced with a backward and forward difference each. A similar thing happens here: the update equations are no longer parallel, since $x[k+1]$ (first integrator, BP output) is used in the computation of $y[k+1]$ (second integrator, LP output).

It's probably best to first define what *symplectic structure* means, and then to see how it can be preserved in the analog to digital conversion. What I remember is that it is important to be able to factor the update into two successive triangular updates. This is exactly what happens in the mixed FW/BW SVF discretization. At full resonance $k = 0$ (we're talking about lossless systems only) in the update

$$\begin{array}{rcl} zx_1 & = & x_1 - a(kx_1 + x_2 - i) \\ zx_2 & = & x_2 + azx_1 \end{array},$$

we have two consecutive operations which hide behind the presence of zx_1 in the equation (as opposed to x_1). This succession of updates can be represented by two triangular matrices.

$$A = LU, \quad L = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix}.$$

Note that this matrix is not orthogonal or $A^T A \neq I$. However, it is symplectic. A 2D discrete system with feedback matrix A is symplectic if $A^T J A = J$ with

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

This property is satisfied by the conservative forward/backward SVF. Note that $J^2 = -I$.

Because $\det A = 1$ and $A \in R^{2\times 2}$, if the poles are complex conjugate they have to be magnitude 1 so the system is still stable.

So, is a symplectic system always stable? In general, a $A \in R^{2N \times 2N}$ is symplectic if $A^T J_N A = J$ where

$$J_N = \begin{bmatrix} 0 & I \\ -I_N & 0 \end{bmatrix}$$

where I is the $N \times N$ identity matrix. From this we have det$A = 1$. So it is conservative in that state energy is preserved, but that doesn't mean the individual 2D subsystems should be energy–preserving. In [2], p282 it is explained how for each eigenvalue λ, the inverse λ^{-1} is also an eigenvalue.

An example of a 4D symplectic matrix can be constructed from a 2D symplectic matrix $A_1^T J_1 A_1 = J_1$ by constructing the matrix

$$A_2' = \begin{bmatrix} rA_1 & 0 \\ 0 & r^{-1}A_1 \end{bmatrix}.$$

Straightforward computation shows that $A_2'^T J_2' A_2' = J_2'$ where

$$J_2' = \begin{bmatrix} 0 & J_1 \\ -J_1 & 0 \end{bmatrix},$$

which is a permutation of J_2, meaning $J_2' = P^T J_2 P$. From this follows that $A_2 = P^T A_2' P$ is symplectic. Since P is orthogonal, A_2 and A_2' have the same eigenvalues. The eigenvalues of A_1 are unit norm complex conjucate which makes the eigenvalues of A_2 complex conjugate pairs of magnitude r and r^{-1} respectively.